Trinity College

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

(a) The point $P(-3, 5)$ is translated by the column vectors $\begin{bmatrix} 2 \\ -7 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ to $P'(9, -11)$. Determine the values of the constants x and y. (2 marks)

Solution
$\begin{bmatrix} -3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ -7 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -9 \end{bmatrix}$
Specific behaviours
✓ value of x ✓ value of y

(b) Determine the single matrix that represents, in order, the composition of a rotation of $30°$ anti-clockwise about the origin followed by reflection in the line $y = x$. Express matrix coefficients in exact form. (4 marks)

Solution
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
Specific behaviours
✓ matrix for rotation ✓ matrix for reflection ✓ multiplies in correct order ✓ correct matrix

Question 10 **(6 marks)**

An exam has two sections, I and II, containing 7 and 14 questions respectively.

Determine the number of different combinations of questions a candidate could choose if they must answer

(a) 2 questions from Section I and 3 questions from Section II. (2 marks)

Solution
$\binom{7}{2}\binom{14}{3} = 21 \times 364$ $= 7\,644$ ways
Specific behaviours
✓ uses multiplication of combinations ✓ correct number

(b) 5 questions, all chosen from the same section. (2 marks)

Solution
$\binom{7}{0}\binom{14}{5} + \binom{7}{5}\binom{14}{0} = 2002 + 21$ $= 2\,023$ ways
Specific behaviours
✓ uses addition of combinations ✓ correct number

(c) 5 questions, with at least one question from each section. (2 marks)

Solution
$\binom{21}{5} = 20349$ $20349 - 2023 = 18\,326$ ways
Specific behaviours
✓ calculates total ways, no restriction ✓ subtracts answer from (b)

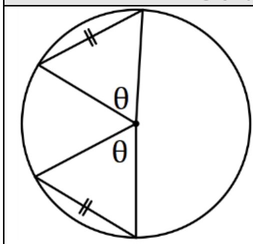
Question 11 **(6 marks)**

(a) A circle property says that if chords of a circle are of equal length then they subtend equal angles at the centre.

(i) Write the converse of this statement. (1 mark)

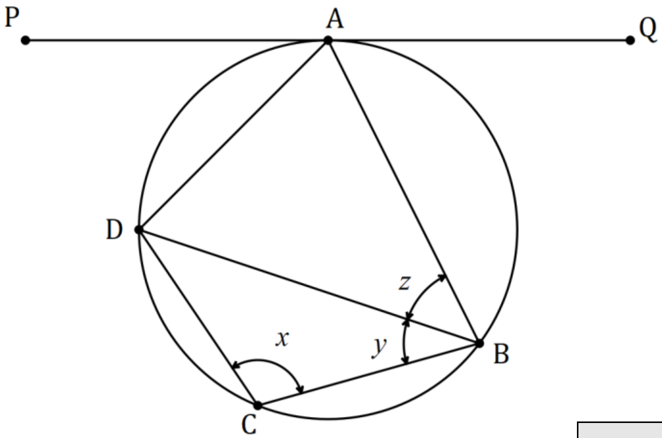
Solution
If chords of a circle subtend equal angles at the centre then they are of equal length.
Specific behaviours
✓ writes converse

(ii) Draw a diagram to illustrate the converse statement and state whether the converse is also true. (2 marks)

Solution
 Converse is true.
Specific behaviours
✓ diagram ✓ states converse true

(b) The diagram below shows four points A, B, C and D lying on the circumference of a circle. The line PQ is a tangent to the circle at A, $\angle BDC = 21°$, $\angle PAD = 35°$ and $\angle QAB = 62°$.

Determine the size of angles x, y and z. (3 marks)



Solution
$x = 180 - (180 - 35 - 62) = 97°$ $y = 180 - 97 - 21 = 62°$ $z = \angle PAD = 35°$
Specific behaviours
✓✓✓ each angle

Question 12 **(9 marks)**

(a) If $\mathbf{p} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{q} = 3\mathbf{i} + 2\mathbf{j}$ determine

(i) the angle between the directions of $\mathbf{p}$ and $\mathbf{q}$, to the nearest tenth of a degree. (2 marks)

Solution
Using CAS angle is $60.255 \approx 60.3°$ (1dp)
Specific behaviours
✓ states angle ✓ rounds correctly

(ii) the scalar projection of $\mathbf{q}$ on $\mathbf{p}$. (2 marks)

Solution
$\sqrt{3^2 + 2^2} \times \cos(60.255)$ ≈ 1.79 Exact: $\frac{4\sqrt{5}}{5}$
Specific behaviours
✓ expression for projection ✓ states value

(b) The vector $21\mathbf{i} + 5a\mathbf{j}$ has a magnitude of 29 and is perpendicular to the vector $4\mathbf{i} - 2b\mathbf{j}$. Determine the values of the constants a and b, where $a < b$. (5 marks)

Solution
$21^2 + (5a)^2 = 29^2$ $a = \pm 4$ $(21)(4) + (5a)(-2b) = 0$ $84 - 10(\pm 4)b = 0$ $b = \pm \frac{21}{10}$ $a = -4, \qquad b = -\frac{21}{10}$
Specific behaviours
✓ uses magnitude to form equation ✓ calculate values of a ✓ uses dot product to form equation ✓ calculate values of b ✓ chooses correct pairing

Question 13 **(6 marks)**

(a) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$. (4 marks)

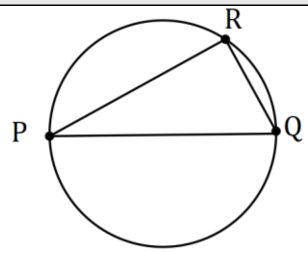
Solution
$LHS = \sin 3A$ $= \sin(A + 2A)$ $= \sin A \cos 2A + \cos A \sin 2A$ $= \sin A\,(1 - 2\sin^2 A) + \cos A\,(2\sin A \cos A)$ $= \sin A - 2\sin^3 A + 2\sin A\,(1 - \sin^2 A)$ $= 3\sin A - 4\sin^3 A$ $= RHS$
Specific behaviours
✓ expands sum of A and 2A ✓ uses double angle identity for cos2A ✓ uses double angle identity for sin2A ✓ expands and simplifies

(b) Hence, or otherwise, solve $3\sin A - 4\sin^3 A = \frac{1}{2}, 0 \le A \le \frac{\pi}{3}$. (2 marks)

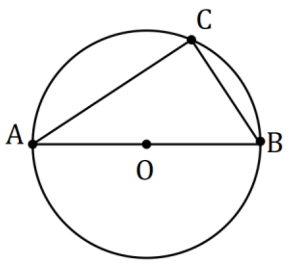
Solution
$\sin 3A = \frac{1}{2}$ $3A = \frac{\pi}{6}, \frac{5\pi}{6}$ $A = \frac{\pi}{18}, \frac{5\pi}{18}$
Specific behaviours
✓ one correct solution ✓ all solutions within required domain

Question 14 **(7 marks)**

(a) Point R lies on the circumference of a circle with diameter $PQ = 51$ cm, so that $PR = 4RQ$. Determine the exact length RQ. (3 marks)

Solution

Let $RQ = x$, then $x^2 + (4x)^2 = 51^2$
$x = 3\sqrt{17}$ cm
Specific behaviours
✓ diagram ✓ uses Pythagoras' Theorem ✓ states length

(b) Use a vector method to prove that the angle in a semi-circle is a right-angle. (4 marks)

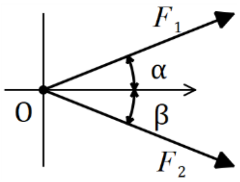


Let $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OB} = \mathbf{b}$.

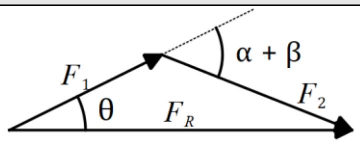
Solution
$\overrightarrow{AC} = \mathbf{b} + \mathbf{c}$
$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$
$\overrightarrow{AC} \cdot \overrightarrow{BC} = (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{b})$
$= \lvert\mathbf{c}\rvert^2 - \lvert\mathbf{b}\rvert^2$
$= 0$, as $\lvert\mathbf{c}\rvert = \lvert\mathbf{b}\rvert = \text{radius}$
Hence $\angle ACB = 90°$
Specific behaviours
✓ vectors for AC, BC ✓ forms scalar product ✓ simplifies scalar product, with reasons ✓ concludes angle is right

Question 15 **(8 marks)**

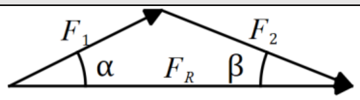
In the diagram below, forces F_1 and F_2 act on a body at the origin.



(a) If $F_1 = 50\ N$, $F_2 = 65\ N$, $\alpha = 22°$ and $\beta = 32°$, determine the magnitude of the resultant force and the angle it makes with the positive x axis. (5 marks)

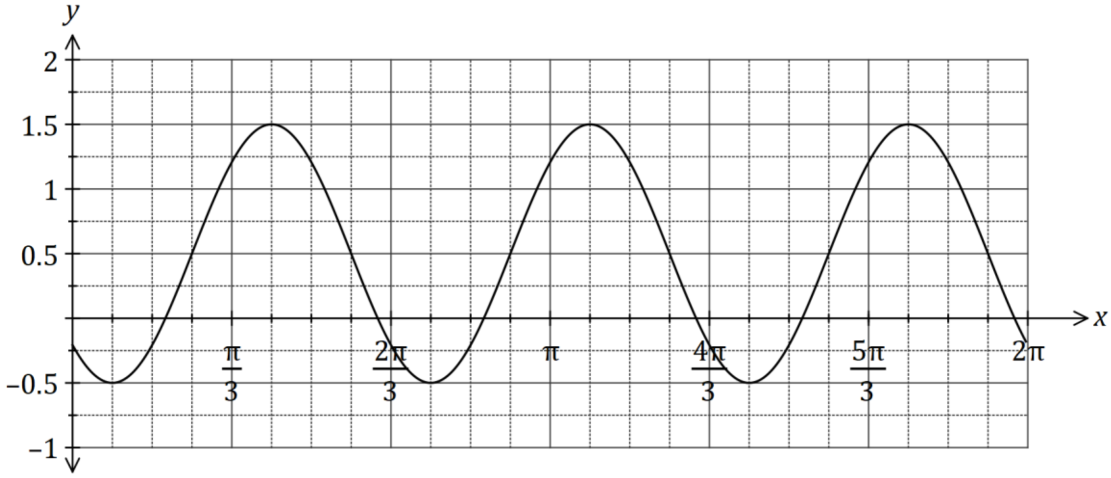
Solution
 $F_R = \sqrt{50^2 + 65^2 - 2(50)(65)\cos 126}$ $F_R = 102.7\ N$ $\frac{\sin\theta}{65} = \frac{\sin 126}{102.7}$ $\theta = 30.8°$ $\phi = 22° - 30.8° = -8.8°$
Specific behaviours
✓ sketch with forces nose to tail   ✓ uses cosine rule for magnitude   ✓ states magnitude   ✓ uses sine rule for angle   ✓ states angle with x axis

(b) If $F_1 = 75\ N$ and $F_2 = 95\ N$, determine the angles α and β so that the resultant force is directed along the positive x axis and has a magnitude of $155\ N$. (3 marks)

Solution
 $\alpha = \cos^{-1}\left(\frac{75^2 + 155^2 - 95^2}{2(75)(155)}\right)$ $\alpha = 27.5°$ $\beta = \cos^{-1}\left(\frac{95^2 + 155^2 - 75^2}{2(95)(155)}\right)$ $\beta = 21.4°$
Specific behaviours
✓ sketch with resultant on x axis   ✓ uses cosine rule for angle   ✓ uses sine or cosine rule for second angle

Question 16 **(10 marks)**

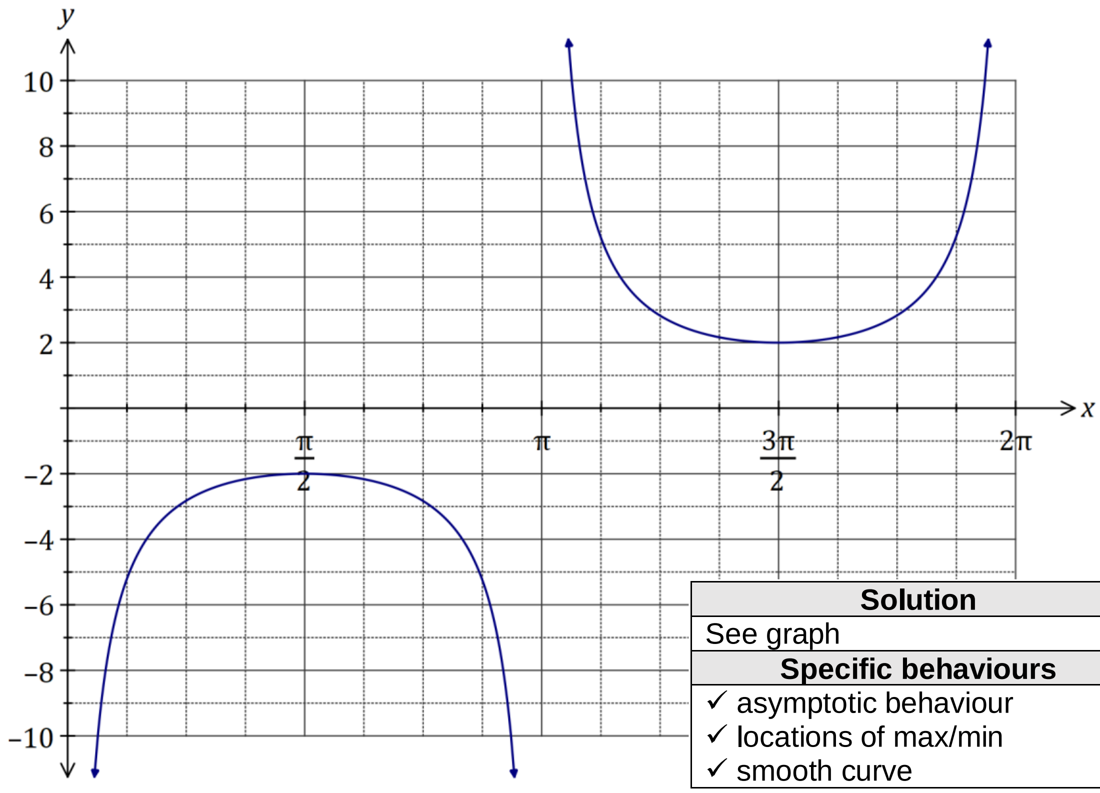
(a) The graph of $y = \cos\big(a(x+b)\big) + c$ is shown below for $0 \le x \le 2\pi$.



Determine the value of the positive constants a, b and c. (3 marks)

Solution
$a = 3,\quad b = \frac{\pi}{3} - \frac{\pi}{12} = \frac{\pi}{4},\quad c = \frac{1}{2}$
Specific behaviours
✓ value of a, ✓ value of b, ✓ value of c

(b) On the axes below, sketch the graph of $y = 2\,\mathrm{cosec}(x - \pi)$, $0 \le x \le 2\pi$. (3 marks)



Solution
See graph
Specific behaviours
✓ asymptotic behaviour ✓ locations of max/min ✓ smooth curve

(c) The displacement, x cm, of a particle from a fixed point O varies with time, t seconds, according to the model $x = 4\sin(3\pi t) - 7\cos(3\pi t)$, $t \geq 0$. Determine

(i) the initial displacement of the particle from O. (1 mark)

Solution
$x = -7$
Specific behaviours
✓ correct value

(ii) the exact amplitude of the motion. (1 mark)

Solution
$A = \sqrt{4^2 + 7^2} = \sqrt{65}$ cm
Specific behaviours
✓ correct value

(iii) the period of motion. (1 mark)

Solution
$P = \dfrac{2\pi}{3\pi} = \dfrac{2}{3}$ s
Specific behaviours
✓ correct value

(iv) the first time that the particle passes through O, rounded to two decimal places. (1 mark)

Solution
$t = 0.11$ s
Specific behaviours
✓ correct value

Question 17 **(9 marks)**

Triangle ABC has vertices $A(1,2)$, $B(4,-1)$ and $C(5,3)$.

(a) The vertices ABC are transformed to $A'B'C'$ using matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Write down the new coordinates of the vertices and describe the transformation. (4 marks)

Solution
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 5 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$ $A'(-1,2), B'(-4,-1), C'(-5,3)$ Transformation is a reflection in the line $x = 0$.
Specific behaviours
✓ matrix product   ✓ writes coordinates of vertices   ✓ states reflection   ✓ states equation of line of reflection

(b) The vertices ABC are transformed to $A''B''C''$ by matrix M so that the new coordinates of the vertices are $A''(-4,3)$, $B''(2,12)$ and $C''(-6,15)$.

(i) Determine the transformation matrix M. (3 marks)

Solution
$M\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & 12 \end{bmatrix}$ $M = \begin{bmatrix} -4 & 2 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}^{-1}$ $M = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}$
Specific behaviours
✓ writes matrix equation   ✓ post-multiplies by inverse   ✓ determines M

(ii) If the area of triangle ABC is k square units, express the area of triangle $A''B''C''$ in terms of k. (2 marks)

Solution
$\lvert M \rvert = 6$ $\text{New area} = 6k$
Specific behaviours
✓ determinant of M ✓ expresses area

Question 18 **(7 marks)**

(a) How many numbers must be chosen from the set of integers between 1 and 2017 inclusive to be certain that one of the numbers chosen is a multiple of 10. (3 marks)

Solution
201 multiples of 10 between 1 to 2017. Require $2017 - 201 = 1816$ pigeonholes. Hence must choose $1\,817$ integers.
Specific behaviours
✓ states # of multiples in set ✓ one pigeonhole for every non-multiple ✓ uses pigeonhole principle to add one

(b) A number is formed using four different digits chosen from those in the number 23 814. Determine how many different numbers can be formed that are

(i) even. (1 mark)

Solution
$n(A) = 3 \times 4 \times 3 \times 2 = 72$
Specific behaviours
✓ states number

(ii) greater than 8 000. (1 mark)

Solution
$n(B) = 1 \times 4 \times 3 \times 2 = 24$
Specific behaviours
✓ states number

(iii) even or greater than 8 000. (2 marks)

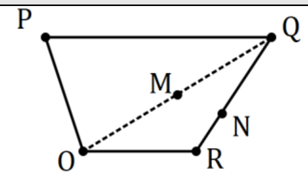
Solution
$n(A \cap B) = 1 \times 2 \times 3 \times 2 = 12$ $n(A \cup B) = 72 + 24 - 12 = 84$ 84 numbers
Specific behaviours
✓ calculates number even and greater than 8 000 ✓ states number

Question 19 **(10 marks)**

(a) Trapezium $OPQR$ has parallel sides PQ and OR. M is the midpoint of OQ and N lies on QR so that $RN:NQ = 1:2$.

Given that $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$ and $\overrightarrow{PQ} = 2\mathbf{r}$, determine the following in terms of $\mathbf{p}$ and $\mathbf{r}$.

(i) $\overrightarrow{OM}$. (2 marks)

Solution

$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{PQ}\right)$ $= \frac{1}{2}(\mathbf{p} + 2\mathbf{r})$ $= \frac{1}{2}\mathbf{p} + \mathbf{r}$
Specific behaviours
✓ indicates half of OQ ✓ correct vector

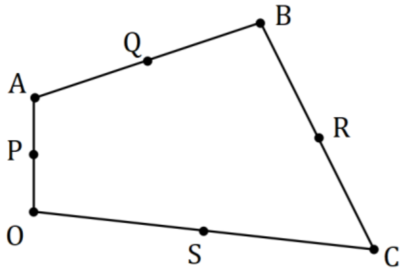
(ii) $\overrightarrow{ON}$. (2 marks)

Solution
$\overrightarrow{ON} = \overrightarrow{OR} + \frac{1}{3}\overrightarrow{RQ}$ $= \overrightarrow{OR} + \frac{1}{3}\left(\overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PQ}\right)$ $= \mathbf{r} + \frac{1}{3}(-\mathbf{r} + \mathbf{p} + 2\mathbf{r})$ $= \frac{4}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$
Specific behaviours
✓ indicates OR and third of RQ ✓ correct vector

(iii) $\overrightarrow{NM}$. (2 marks)

Solution
$\overrightarrow{NM} = \overrightarrow{OM} - \overrightarrow{ON}$ $= \left(\frac{1}{2}\mathbf{p} + \mathbf{r}\right) - \left(\frac{4}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}\right)$ $= \frac{1}{6}\mathbf{p} - \frac{1}{3}\mathbf{r}$
Specific behaviours
✓ indicates difference of (i) and (ii) ✓ correct vector

(b) Quadrilateral $OABC$ is shown below, where P, Q, R and S are the midpoints of the sides OA, AB, BC and OC respectively. Let $\overrightarrow{OP} = \mathbf{a}$, $\overrightarrow{AQ} = \mathbf{b}$ and $\overrightarrow{OS} = \mathbf{c}$.



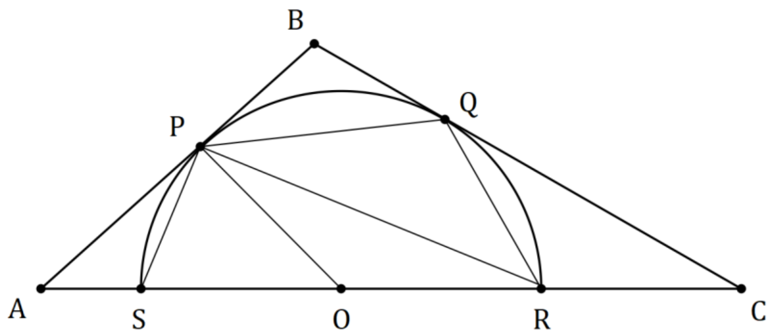
Show that $\overrightarrow{PS} = \overrightarrow{QR}$. (4 marks)

Solution
$\overrightarrow{PS} = \mathbf{c} - \mathbf{a}$ $\overrightarrow{OR} = 2\mathbf{a} + 2\mathbf{b} + \frac{1}{2}\overrightarrow{BC}$ $= 2\mathbf{a} + 2\mathbf{b} + \frac{1}{2}(-2\mathbf{a} - 2\mathbf{b} + 2\mathbf{c})$ $= \mathbf{a} + \mathbf{b} + \mathbf{c}$ $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$ $= \mathbf{a} + \mathbf{b} + \mathbf{c} - (2\mathbf{a} + \mathbf{b})$ $= \mathbf{c} - \mathbf{a}$ $= \overrightarrow{PS}$
Specific behaviours
✓ vector $\overrightarrow{PS}$ ✓ vector $\frac{1}{2}\overrightarrow{BC}$ ✓ vector $\overrightarrow{OR}$ ✓ vector $\overrightarrow{QR}$

Question 20 **(6 marks)**

The diagram shows a semi-circle, with diameter SR and centre O, circumscribed by triangle ABC, in which $\angle BAC = 42°$ and $\angle BCA = 32°$.

Determine, with reasons, the size of angles $\angle PRO$ and $\angle PRQ$.



Solution
$\angle AOP = 90 - 42 = 48°$ ($\angle APO = 90°$, tangent-radius) $\angle PRO = 48 \div 2 = 24°$ (centre $\angle$=2×circumference $\angle$) $\angle COQ = 90 - 32 = 58°$ (as $\angle AOP$) $\angle ORQ = (180 - 58) \div 2 = 61$ (ΔOQR isosceles) $\angle PRQ = 61 - 24 = 37°$
Specific behaviours
✓ $\angle AOP$ ✓ $\angle PRO$ ✓ reasoning ✓ $\angle ORQ$ ✓ $\angle PRQ$ ✓ reasoning

Question 21 **(8 marks)**

The sum of the first n terms of the sequence $2 + 8 + 14 + 20 + \cdots + (6n - 4)$ is $n(3n - 1)$.

(a) Show that this statement is true when $n = 5$. (2 marks)

Solution
$LHS = 2 + 8 + 14 + 20 + 26 = 70$ $RHS = 5(3(5) - 1) = 5 \times 14 = 70$ Hence statement true
Specific behaviours
✓ shows sum of terms for LHS ✓ shows substitution in RHS and states true

(b) Use mathematical induction to prove the statement is true for $n \in \mathbb{Z}, n \geq 5$. (6 marks)

Solution
Assume statement true when $n = k$: $2 + 8 + 14 + 20 + \cdots + (6k - 4) = k(3k - 1)$ When $n = k + 1$: $2 + 8 + 14 + 20 + \cdots + (6k - 4) + (6k - 4 + 6) = k(3k - 1) + (6k - 4 + 6)$ $= 3k^2 + 5k + 2$ $= (k + 1)(3k + 2)$ $= (k + 1)(3(k + 1) - 1)$ $= n(3n - 1)$ when $n = k + 1$ The statement is true for $n = 5$ and by induction, the truth when $n = k$ implies the truth when $n = k + 1$ and hence the statement is true for $n \geq 5$.
Specific behaviours
✓ assumed true for $n = k$ ✓ adds next term to both sides ✓ simplifies RHS ✓ factors k+1 out of RHS ✓ indicates true for $n = k + 1$ ✓ summary statement including truth of $n = 5$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________